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Stability of large- and small-amplitude solitary waves in the generalized Korteweg–de Vries and Euler–Korteweg/Boussinesq equations

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ABSTRACT

This paper establishes that solitary waves for the generalized Korteweg–de Vries equation and for the generalized Boussinesq equation are stable if the flux function p satisfies

$$p'' > 0 \quad \text{and} \quad p''' \leqslant 0.$$

While $p'' > 0$ alone suffices for the stability of waves of sufficiently small amplitude, obvious examples show that $p''' \leqslant 0$ cannot be omitted in the general case.

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0. Introduction

While the stability of solitary waves for the generalized Boussinesq equation

$$V_t - U_x = 0,$$
$$U_t + p(V)_x = -\kappa V_{xxx} \quad \text{with } p' < 0$$

and the generalized Korteweg–de Vries equation

$$V_t + p(V)_x = -\kappa V_{xxx}$$

has been studied in great completeness for the case that

$$p''(V) = V^{q-2}$$

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with some integer $q - 2 \geqslant 0$, the case of more general p seems to have escaped attention. It is however interesting from the point of view of applications. In particular, the generalized Boussinesq equation with

$$p(V) = kV^{-\gamma} \quad \text{with } \gamma \geqslant 1,\ k > 0 \tag{1}$$

describes the flow of an inviscid isothermal ideal (barotropic) fluid with capillarity. In this paper, we mainly establish the following new stability results:

Theorem 1. *Consider the generalized Korteweg–de Vries (gKdV) equation with a smooth function p satisfying $p'' > 0$ and $p''' \leqslant 0$. Then any solitary wave is stable.*

Theorem 2. *Consider the generalized Boussinesq equation with $p : \mathbb{R} \to \mathbb{R}$ or $p : (0, \infty) \to \mathbb{R}$ satisfying $p' < 0$, $p'' > 0$ and $p''' \leqslant 0$. Then any solitary wave is stable.*

These results complement the findings of [4] and [3], respectively; the only overlap of Theorems 1 and 2 with those in [4,3] consisting exactly of the quadratic nonlinearity $p''' \equiv 0$. Note, however, that Theorems 1 and 2 are not restricted to pure power laws. Notably, our result implies stability of all corresponding solitary waves with p satisfying (1).

With no assumptions on the third derivative of p we obtain:

Theorem 3. *Consider the gKdV equation and assume that $p''(v_*) > 0$ for some fixed state $v_* \in \mathbb{R}$. Then there exists an $\varepsilon_1 > 0$ such that for all $\varepsilon \in (0, \varepsilon_1)$ there exists a small amplitude solitary wave of speed $p'(v_*) + \varepsilon$ and this wave is orbitally stable.*

We prove stability of small amplitude solitary waves for a generalization of the Boussinesq equation with variable capillarity $\kappa(v)$, the so-called Euler–Korteweg equation:

Theorem 4. *Consider the Euler–Korteweg equation with $p'(v_*) < 0$ and $p''(v_*) > 0$ for some fixed state $v_* \in \mathbb{R}$. Then:*

(i) *There is a $c_1 < c_* := \sqrt{-p'(v_*)}$ such that for each $c \in (c_1, c_*)$ there exists a small amplitude solitary wave solution with profile (v_c, u_c). The map $c \mapsto (v_c, u_c)$ is $C^\infty((c_1, c_*))$.*
(ii) *There is a $c_2 \in (c_1, c_*)$ such that for each $c \in (c_2, c_*)$ the solitary wave (v_c, u_c) is orbitally stable.*

Theorems 1 to 4 will be proved in Sections 1 to 4. Sections 3 and 4 also contain Theorems 3a and 4a which provide some stability/instability findings in situations which are similar to, but different from those considered in Theorems 3 and 4. Finally, Section 5 presents obvious examples which show that strict convexity of p alone cannot preclude instability for general amplitudes.

1. Stability of large-amplitude waves in the gKdV equation

Consider the generalized Korteweg–de Vries (gKdV) equation

$$V_t + p(V)_x = -\kappa V_{xxx} \tag{2}$$

with

$$p'' > 0 \quad \text{and} \quad p''' \leqslant 0, \tag{3}$$

and a positive constant $\kappa > 0$ which we w.l.o.g. from now on assume to equal one. We are interested in solitary wave solutions, i.e. (nonconstant) traveling wave solutions

$$V(x,t) = v(x-ct) \quad \text{with} \lim_{\xi\to\infty} v(\xi) = \lim_{\xi\to-\infty} v(\xi).$$

Obviously, v is a solitary wave if and only if it is a homoclinic orbit of the ordinary differential equation

$$-cv' + p(v)' = -v'''.$$

With $v_* := \lim_{\xi\to\infty} v(\xi)$, this is equivalent to

$$v'' = cv - cv_* - p(v) + p(v_*). \tag{4}$$

Define f (up to a constant) by the relation

$$-\frac{df(v)}{dv} = p(v)$$

and let

$$F(v,c) := -\frac{1}{2}c(v-v_*)^2 - f(v) + f(v_*) - p(v_*)(v-v_*).$$

Then (4) reads

$$v'' = -\frac{\partial F(v,c)}{\partial v};$$

a first integral is given by

$$I\big(v,v'\big) = \frac{1}{2}v'^2 + F(v,c). \tag{5}$$

Lemma 1. *Consider* (2) *with* p *satisfying* (3) *and fix an arbitrary state* $v_* \in \mathbb{R}$. *There is a smooth bijection* $v_m : (p'(v_*), p'(\infty)) \to (v_*, \infty)$ *such that the following holds. A solitary wave homoclinic to* v_* *and of speed* c *exists if and only if* $p'(v_*) < c < p'(\infty)$ *and* $v_m(c) > v_*$.

Proof. As

$$\frac{\partial^2 F}{\partial v^2}(v_*,c) = -c + p'(v_*),$$

$(v_*, 0)$ is a saddle point for

$$v' = w,$$
$$w' = -\frac{\partial F(v,c)}{\partial v}$$

if and only if $c > p'(v_*)$. Consider now only such c. In view of

$$\frac{\partial F}{\partial v}(v,c) = -c(v - v_*) + p(v) - p(v_*)$$

the following is immediate. In any case,

$$F(v,c) < 0 \quad \text{for all } v < v_*;$$

if $c \geqslant p'(\infty)$, then also $F(v,c) < 0$ for all $v > v_*$; if however $c < p'(\infty)$, then

$$F\big(v_m(c), c\big) = 0$$

for a unique value $v_m(c)$. $\square$

Definition 1. A traveling wave v of (2) is called orbitally stable if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that for any solution $V \in v + C([0,\infty); H^s(\mathbb{R}))$, $s > 3/2$, of (2), closeness at initial time,

$$\big\| V(\cdot, 0) - v(\cdot) \big\|_{H^1(\mathbb{R})} < \delta$$

implies closeness at any time

$$\inf_{\sigma \in \mathbb{R}} \big\| V(\cdot, t) - v(\cdot + \sigma) \big\|_{H^1(\mathbb{R})} < \varepsilon \quad \text{for all } t > 0.$$

Recall that the existence time of any solution $V \in C([0,T), H^s(\mathbb{R}))$ with $s > 3/2$ that has a uniform H^1 bound is ∞ (Theorem 2.1 of [4]).

The main result of this section is

Theorem 1. *Consider* (2) *with a smooth function p satisfying* (3)*. Then any solitary wave v for* (2) *is stable.*

Proof. In order to show nonlinear orbital stability, we apply a technique that was developed in [6] and first used for the gKdV equation in [4]; cf. also [9]. The stability of solitary waves is decided by the convexity behavior of so-called *moment of instability*. To recapitulate this very briefly, recall first that Eq. (2) has the Hamiltonian form

$$V_t = \mathcal{J}\delta\mathcal{H}[V] \quad \text{with } \mathcal{H}[V] = \int H(V)$$

where

$$H(V) = \frac{1}{2}(V_x)^2 + f(V) - f(v_*) + p(v_*)(V - v_*), \quad \text{and} \quad \mathcal{J} = \partial_x.$$

Introduce the functional

$$\mathcal{I}[V] = \int I(V) \quad \text{with } I(V) = \frac{1}{2}(V - v_*)^2$$

and denote for the moment the traveling wave by Φ_c. Then the profile equation (4) reads

$$\mathcal{J}\delta\big(\mathcal{H}[\Phi_c] + c\mathcal{I}[\Phi_c]\big) = 0.$$

The following theorem is due to [4,9]:

Theorem (∗). *Let $X = H^1(\mathbb{R})$. Assume*

(A1) (*Local well-posedness.*) *For any $\mu > 0$, there exists $T > 0$ such that for all $U_0 \in X$ with $U_{0X} \leqslant \mu$, the system $U_t = \mathcal{J}\delta\mathcal{H}[U]$ has a solution defined at least on $[0, T)$ such that $U(0) = U_0$. U satisfies $\mathcal{H}[U(t)] = \mathcal{H}[U_0]$ and $\mathcal{I}[U(t)] = \mathcal{I}[U_0]$ for all $t \in [0, T)$.*

(A2) (*Existence of solitary waves.*) *There are an interval (c_-, c_+) and a mapping*

$$(c_-, c_+) \to X$$

$$c \mapsto \Phi_c$$

such that for each $c \in (c_-, c_+)$,

(i) $\delta(\mathcal{H} + c\mathcal{I})[\Phi_c] = 0$,

(ii) $\Phi_c'' \in X$, *and*

(iii) $\Phi_c' \neq 0$.

(A3) (*Spectrum of the Hessian.*) *For each $c \in (c_-, c_+)$, the operator $L_c = \delta^2(\mathcal{H} + c\mathcal{I})[\Phi_c]$ has precisely one negative eigenvalue which is simple and its kernel is spanned by (Φ_c'). The rest of its spectrum is positive and bounded away from zero.*

Then, the solitary wave with profile Φ_c is stable if and only if its moment of instability

$$m(c) := (\mathcal{H} + c\mathcal{I})(\Phi_c)$$

satisfies

$$\frac{d^2 m}{dc^2}(c) > 0. \tag{6}$$

For gKdV, local well-posedness (A1) is due to Kato [7]. Assumption (A3) has been proved by Pego and Weinstein [9]. As (A2) is obvious in our case we will have proved Theorem 1 once we have shown the convexity of $m(c)$. For our case,

$$\begin{aligned} m(c) &= \int_{-\infty}^{\infty} \frac{1}{2}v'^2 + f(v) - f(v_*) + p(v_*)(v - v_*) + \frac{c}{2}(v - v_*)^2 \, dx \\ &= \int_{-\infty}^{\infty} v'^2 \, dx = 2\int_{v_*}^{v_m(c)} v' \, dv = 2\int_{v_*}^{v_m(c)} \big(-2F(v, c)\big)^{1/2} \, dv \\ &= 4\int_0^{(v_m(c)-v_*)^{1/2}} \big(-2F\big(v_m(c) - u^2, c\big)\big)^{1/2} u \, du, \quad \text{where } u := \big(v_m(c) - v\big)^{1/2}. \end{aligned}$$

Differentiating, we obtain

$$m'(c) = 2v_m'(c)\big(-2F(v_*, c)\big)^{1/2} + 4\int_0^{(v_m(c)-v_*)^{1/2}} \frac{d}{dc}\big(\big(-2F\big(v_m(c) - u^2, c\big)\big)^{1/2} u\big) \, du$$

$$= 4 \int_0^{(v_m(c)-v_*)^{1/2}} -\frac{F_v(v_m(c)-u^2, c)v_m'(c) + F_c(v_m(c)-u^2, c)}{(-2F(v_m(c)-u^2, c))^{1/2}} u\,du$$

$$= 2v_m'(c) \int_{v_*}^{v_m(c)} \frac{\partial}{\partial v}\bigl(\bigl(-2F(v,c)\bigr)^{1/2}\bigr)\,dv - 4 \int_0^{(v_m(c)-v_*)^{1/2}} \frac{F_c(v_m(c)-u^2, c)}{(-2F(v_m(c)-u^2, c))^{1/2}} u\,du$$

and thus

$$m''(c) = \int_{v_*}^{v_m(c)} \frac{-4F(v,c)(v-v_*)v_m'(c) + (v-v_*)^2(F_v(v,c)v_m'(c) + F_c(v,c))}{(-2F(v,c))^{3/2}}\,dv. \tag{7}$$

Obviously, the following proposition will finish the proof of the theorem.

Proposition 1. *For fixed c, define $T(v)$ as*

$$T(v) := -4F(v,c)v_m'(c) + (v-v_*)\bigl(F_v(v,c)v_m'(c) + F_c(v,c)\bigr).$$

Then $T(v) > 0$ for all $v \in (v, v_m(c))$.

Proof. We introduce two smooth functions α_1, α_2 via

$$f(v) = f(v_*) + f'(v_*)(v-v_*) + \frac{\alpha_1(v)}{2}(v-v_*)^2 \quad \text{and}$$
$$p(v) = p(v_*) - \alpha_2(v)(v-v_*)$$

and numbers α_3, α_4 via

$$\alpha_3 := \frac{-p(v_m(c)) + p(v_*)}{v_m(c) - v_*} = \alpha_2\bigl(v_m(c)\bigr),$$
$$\alpha_4 := \frac{-p(\bar{v}_*(c)) + p(v_*)}{\bar{v}_*(c) - v_*} = \alpha_2\bigl(\bar{v}_*(c)\bigr);$$

here $\bar{v}_*(c)$ denotes the point in $(v_*, v_m(c))$ where $F_v(v,c)$ vanishes. By construction $c = -\alpha_4$ and

$$F(v,c) = \frac{(v-v_*)^2}{2}\bigl(\alpha_4 - \alpha_1(v)\bigr),$$
$$F_v(v,c) = (v-v_*)\bigl(\alpha_4 - \alpha_2(v)\bigr),$$
$$F_c(v,c) = -\frac{1}{2}(v-v_*)^2.$$

Finally, as for all c

$$F\bigl(v_m(c), c\bigr) \equiv 0,$$

it follows that

$$v_m'(c) = \frac{(v_m(c) - v_*)}{-2(\alpha_3 + c)}.$$

It remains to show that

$$Q(v) := (v - v_*)(\alpha_3 - \alpha_4) + (v_m(c) - v_*)\big(2\alpha_1(v) - \alpha_4 - \alpha_2(v)\big) \geqslant 0. \qquad \square$$

The following lemma establishes properties of the α_i:

Lemma 2. *Let α_i defined as above. Then we have for all $v \in (v_*, v_m(c))$*

(i) $\alpha_3 \leqslant \alpha_4$,
(ii) $\alpha_3 \leqslant \alpha_2(v)$,
(iii) $\alpha_4 \leqslant \alpha_1(v)$,
(iv) $\alpha_2(v) \leqslant \alpha_1(v)$.

Proof. The proof of the lemma uses the convexity of p in many places. Relations (i) and (ii) can be seen easily comparing the slope of the secant line connecting v_* and v_m (which corresponds to α_3) with those connecting interior points (corresponding to $\alpha_2(v)$ and α_4 resp.).

To show relation (iii) recall that $F(v,c) \leqslant 0$ between v_* and $v_m(c)$ and so we see from

$$F(v,c) = \frac{1}{2}(v - v_*)^2\big(\alpha_4 - \alpha_1(v)\big)$$

that $\alpha_4 \leqslant \alpha_1(v)$.

The last relation (iv) follows from the following observation: We have

$$\begin{aligned}\alpha_1(v) - \alpha_2(v) &= \frac{1}{(v - v_*)^2}\big\{2f(v) - 2f(v_*) + p(v_*)(v - v_*) + p(v)(v - v_*)\big\} \\ &= \frac{2}{(v - v_*)}\left\{-\frac{1}{(v - v_*)}\int_{v_*}^{v} p(s)\,ds + \frac{p(v_*) + p(v)}{2}\right\} > 0. \qquad \square\end{aligned}$$

Lemma 3. *$Q(v)$ satisfies $Q(v_m(c)) \geqslant 0$.*

Proof. By definition, we have $\alpha_2(v_m(c)) = \alpha_3$ which yields

$$Q\big(v_m(c)\big) = \big(v_m(c) - v_*\big)\big(\alpha_3 - 2\alpha_4 + 2\alpha_1\big(v_m(c)\big) - \alpha_2\big(v_m(c)\big)\big) \geqslant 0$$

by Lemma 2(iii). $\square$

Lemma 4. *$Q'(v) < 0$ for all $v \in (v_*, v_m(c))$.*

Proof. As

$$\alpha_1'(v) = \frac{1}{v - v_*}\big(2\alpha_2(v) - 2\alpha_1(v)\big) \quad \text{and}$$
$$\alpha_2'(v) = \frac{1}{v - v_*}\big(-p'(v) - \alpha_2(v)\big)$$

we obtain

$$Q'(v) = \alpha_3 - \alpha_4 + \frac{v_m(c) - v_*}{v - v_*}\big(5\alpha_2(v) - 4\alpha_1(v) + p'(v)\big).$$

Since $\alpha_3 - \alpha_4 \leqslant 0$ it is sufficient to show that for $v > v_*$

$$5\alpha_2(v) - 4\alpha_1(v) + p'(v) < 0$$

or, equivalently

$$A(v) := -8f(v) + 8f(v_*) - 5p(v)(v - v_*) - 3p(v_*)(v - v_*) + p'(v)(v - v_*)^2 < 0.$$

This however is true as $A(v_*) = 0$ and $A'(v) \leqslant 0$ for $v > v_*$. To see the latter expand p as a function of v_*:

$$p(v_*) = p(v) - p'(v)(v - v_*) + \frac{1}{2}p''(\tau)(v - v_*)^2 \quad \text{where } \tau \in (v_*, v).$$

Then

$$\begin{aligned} A'(v) &= 3p(v) - 3p(v_*) - 3p'(v)(v - v_*) + p''(v)(v - v_*)^2 \\ &= \left(p''(v) - \frac{3}{2}p''(\tau)\right)(v - v_*)^2 \\ &< \big(p''(v) - p''(\tau)\big)(v - v_*)^2 \leqslant 0 \end{aligned}$$

since $p''' \leqslant 0$. □

Lemmata 2 and 3 imply that Q, and thus T are positive. □

Remark. Replacing V with $-V$ shows that Theorem 1 remains true if (3) is replaced by

$$p'' < 0 \quad \text{and} \quad p''' \leqslant 0.$$

2. Stability of large amplitude solitary waves in the generalized Boussinesq equation

Consider the generalized Boussinesq equation

$$\begin{aligned} V_t - U_y &= 0, \\ U_t + p(V)_y &= -V_{yyy} \end{aligned} \tag{8}$$

with a smooth $p : \mathbb{R} \to \mathbb{R}$ or $p : (0, \infty) \to \mathbb{R}$ satisfying $p' < 0$ and

$$p'' > 0 \quad \text{and} \quad p''' \leqslant 0. \tag{9}$$

In contrast to the gKdV equation where the sign of the first derivative is irrelevant we now need p to be strictly decreasing. The profile equation for a solitary wave $(v, u)(\xi)$ connecting $(v_*, 0)$ reads

$$\begin{aligned} v'' &= -c^2 v + c^2 v_* - p(v) + p(v_*) \\ &=: -\frac{\partial F(v, c)}{\partial v} \end{aligned} \tag{10}$$

and admits a first integral given by

$$\begin{aligned} I(v, v') &= \frac{1}{2}v'^2 + F(v, c) \\ &= \frac{1}{2}v'^2 + \frac{1}{2}c^2(v - v_*)^2 - f(v) + f(v_*) - p(v_*)(v - v_*) \end{aligned}$$

with

$$-\frac{df(v)}{dv} = p(v).$$

Lemma 5. *Consider* (8) *with a smooth function* $p : \mathbb{R} \to \mathbb{R}$ *or* $p : (0, \infty) \to \mathbb{R}$ *satisfying* $p' < 0$ *and* (9). *Fix an arbitrary state* v_* *in the interval of definition of* p. *There is a smooth bijection*

$$v_m(c) : \left(\sqrt{-p'(\infty)}, \sqrt{-p'(v_*)}\right) \cup \left(-\sqrt{-p'(v_*)}, -\sqrt{-p'(\infty)}\right) \to (v_*, \infty)$$

such that the following holds: A solitary wave homoclinic to v_* *and of speed* c *exists if and only if* $p'(\infty) < -c^2 < p'(v_*)$ *and* $v_m(c) > v_*$.

Proof. Follows by Lemma 1 and the fact that the profile equation of the gKdV equation with speed c is the profile equation of the generalized Boussinesq equation with speed $-c^2$. In the sequel, we restrict attention to the cases $c > 0$ w.l.o.g. □

Let us recall the definition of orbital stability for the generalized Boussinesq equation.

Definition 2. A traveling wave (v, u) of (8) is called orbitally stable if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that for any solution $(V, U) \in (v, u) + C([0, T]; H^3(\mathbb{R}) \times H^2(\mathbb{R}))$ of (8), closeness at initial time,

$$\left\|(V, U)(\cdot, 0) - (v, u)(\cdot)\right\|_{H^1 \times L^2} < \delta$$

implies closeness at any time

$$\inf_{\sigma \in \mathbb{R}} \left\|(V, U)(\cdot, t) - (v, u)(\cdot + \sigma)\right\|_{H^1 \times L^2} < \varepsilon \quad \text{for all } t > 0.$$

In this case global well-posedness follows from [3]. Our main result in this section is

Theorem 2. *Consider* (8) *with a smooth function* $p : \mathbb{R} \to \mathbb{R}$ *or* $p : (0, \infty) \to \mathbb{R}$ *with* $p' < 0$ *satisfying* (9). *Then any solitary wave* v *for* (8) *is stable.*

Proof. The moment of instability is closely related to the moment of instability of the gKdV equation (cf. also Section 5, Remark 3). Briefly, Eq. (8) can be written as

$$W_t = \mathcal{J}\delta\mathcal{H}[W] \quad \text{with } \mathcal{H}[W] = \int H(W),$$

where

$$W = \begin{pmatrix} V \\ U \end{pmatrix},$$

$$H(W) = \frac{1}{2}(V_y)^2 + \frac{1}{2}U^2 + f(V) - f(v_*) + p(v_*)(V - v_*) \quad \text{and} \quad \mathcal{J} = \begin{pmatrix} 0 & \partial_y \\ \partial_y & 0 \end{pmatrix}.$$

Introducing the functional

$$\mathcal{I}[W] = \int I(W) \quad \text{with } I(W) = U(V - v_*)$$

and denoting the profile by

$$\Phi_c = \begin{pmatrix} v \\ u \end{pmatrix}$$

the profile equation (10) equivalently reads

$$\mathcal{J}\delta\big(\mathcal{H}[\Phi_c] + c\mathcal{I}[\Phi_c]\big) = 0.$$

In this context, Theorem $(*)$ has been established in [3,10] (see also [8]) with $X = H^3(\mathbb{R}) \times H^2(\mathbb{R})$.

Assumptions (A1) and (A3) having been proved in [3], we are again left with showing the convexity of $m(c)$. We obtain

$$\begin{aligned} m(c) &= \int_{-\infty}^{\infty} \big(v'\big)^2\, dx = 2 \int_{v_*}^{v_m(c)} \big(-2F(v,c)\big)^{1/2}\, dv \\ &= 4 \int_0^{(v_m(c)-v_*)^{1/2}} \big(-2F\big(v_m(c) - u^2, c\big)\big)^{1/2} u\, du, \quad \text{with } u := \big(v_m(c) - v\big)^{1/2}. \end{aligned}$$

Differentiating twice yields

$$\frac{m''(c)}{2} = \int_{v_*}^{v_m(c)} \frac{(v - v_*)(2F(v,c)((v - v_*) + 2cv_m'(c)) - c(v - v_*)(F_v(v,c)v_m'(c) + F_c(v,c)))}{(-2F(v,c))^{3/2}}\, dv.$$

Again, we will prove the positivity of the integrand, which will be similar to but slightly more difficult than in the gKdV-case.

Proposition 2. *For fixed c, define $T(v)$ as*

$$T(v) := 2(v - v_*)F(v,c) + 4cv_m'(c)F(v,c) - c(v - v_*)\big(F_v(v,c)v_m'(c) + F_c(v,c)\big).$$

Then $T(v) > 0$ for all $v \in (v_*, v_m(c))$.

Proof. Define functions $\alpha_1(v), \alpha_2(v)$, and numbers α_3 and α_4 as in Proposition 1 with $v_m(c)$ and $\bar{v}_*(c)$ now referring to $F(v,c)$ satisfying (10). Recall Lemma 2. It follows that $c^2 = \alpha_4$ and

$$F(v,c) = \frac{(v-v_*)^2}{2}\big(\alpha_4 - \alpha_1(v)\big),$$
$$F_v(v,c) = (v-v_*)\big(\alpha_4 - \alpha_2(v)\big),$$
$$F_c(v,c) = -\frac{1}{2}(v-v_*)^2.$$

$v_m'(c)$ can now be expressed as

$$v_m'(c) = -\frac{c(v_m(c)-v_*)}{(\alpha_4-\alpha_3)}.$$

After trivial manipulations we see that it suffices to show that

$$Q(v) := (v-v_*)\alpha_1(v)\alpha_3 + (v_m(c)-v)\alpha_1(v)\alpha_4 + (v_m(c)-v_*)\alpha_4\big(\alpha_1(v)-\alpha_2(v)-\alpha_4\big) \geqslant 0.$$

Lemma 6. *$Q(v)$ satisfies $Q(v_m(c)) \geqslant 0$.*

Proof.

$$Q\big(v_m(c)\big) = (\alpha_3+\alpha_4)\big(\alpha_1\big(v_m(c)\big)-\alpha_4\big) \geqslant 0$$

by Lemma 2(iii) and the fact that for the generalized Boussinesq equation all α_i are positive since $p' < 0$. □

Lemma 7. *$Q'(v) < 0$ for all $v \in (v_*, v_m(c))$.*

Proof. Here, a little more than in the gKdV-case has to be done. We have

$$\begin{aligned} Q'(v) &= \alpha_1(v)(\alpha_3-\alpha_4) + \alpha_3\big(2\alpha_2(v)-2\alpha_1(v)\big) \\ &\quad + \frac{v_m(c)-v}{v-v_*}\big(\alpha_4\big(2\alpha_2(v)-2\alpha_1(v)\big)\big) \\ &\quad + \frac{v_m(c)-v_*}{v-v_*}\alpha_4\big(3\alpha_2(v)-2\alpha_1(v)+p'(v)\big). \end{aligned}$$

Obviously by Lemma 2,

$$\alpha_1(v)(\alpha_3-\alpha_4) < \big(\alpha_1(v)-\alpha_2(v)\big)(\alpha_3-\alpha_4)$$

and so

$$\begin{aligned} Q'(v) &< \big(\alpha_1(v)-\alpha_2(v)\big)(\alpha_3-\alpha_4) + \alpha_3\big(2\alpha_2(v)-2\alpha_1(v)\big) \\ &\quad + \frac{v_m(c)-v}{v-v_*}\big(\alpha_4\big(2\alpha_2(v)-2\alpha_1(v)\big)\big) \\ &\quad + \frac{v_m(c)-v_*}{v-v_*}\alpha_4\big(3\alpha_2(v)-2\alpha_1(v)+p'(v)\big). \end{aligned}$$

The proof of the lemma will be finished once we prove that

$$2\big(\alpha_2(v)-\alpha_1(v)\big)\left\{\frac{(v-v_*)}{2}(\alpha_4-\alpha_3)+(v-v_*)\alpha_3+\big(\big(v_m(c)-v\big)+\big(v_m(c)-v_*\big)\big)\alpha_4\right\}$$
$$+\big(v_m(c)-v_*\big)\alpha_4\big(\alpha_2(v)+p'(v)\big)$$

is less or equal to zero. Equivalently

$$-1\geqslant 2\frac{\alpha_2(v)-\alpha_1(v)}{\alpha_2(v)+p'(v)}\frac{\frac{v-v_*}{v_m(c)-v_*}\alpha_3+(\frac{v_m(c)-v}{v_m(c)-v_*}+1)\alpha_4+\frac{v-v_*}{2(v_m(c)-v_*)}(\alpha_4-\alpha_3)}{\alpha_4}.$$

As the third factor can be estimated from below by $\frac{3}{2}$, it remains to show that

$$\frac{\alpha_1(v)-\alpha_2(v)}{\alpha_2(v)+p'(v)}\geqslant\frac{1}{3}$$

or, equivalently

$$B(v):=6f(v)-6f(v_*)+6p(v_*)(v-v_*)+4\big(p(v)-p(v_*)\big)(v-v_*)-p'(v)(v-v_*)^2\geqslant 0.$$

We have $B(v_*)=0$ and with

$$p(v_*)=p(v)-p'(v)(v-v_*)+\frac{1}{2}p''(\tau)(v-v_*)^2\quad\text{with } \tau\in(v_*,v),$$

we obtain

$$\begin{aligned}B'(v)&=-2p(v)+2p(v_*)+2p'(v)(v-v_*)-p''(v)(v-v_*)^2\\&=\big(p''(\tau_2)-p''(v)\big)(v-v_*)^2\geqslant 0.\qquad\square\end{aligned}$$

Again, Lemma 4 and 5 imply that Q and hence T are positive. $\square$

3. (In-)Stability of small amplitude solitary waves in the gKdV equation

Theorem 3. *Consider the gKdV equation and assume that $p''(v_*)>0$ for some fixed state $v_*\in\mathbb{R}$. Then there exists an $\varepsilon_1>0$ such that for all $\varepsilon\in(0,\varepsilon_1)$ there exists a small amplitude solitary wave of speed $p'(v_*)+\varepsilon$ and this wave is orbitally stable.*

Proof. Shifting and scaling v and p, we assume without loss of generality that

$$v_*=0\quad\text{and}\quad p(v)=v^2+O\big(v^3\big)\quad\big(\text{and thus } p'(v_*)=0\big).$$

Let $z:=\varepsilon^{1/2}\xi$. The scaling

$$v(\xi)=:\varepsilon\zeta(z)$$

transforms the equation governing the profile (4) (with speed $c=\varepsilon$) into

$$\ddot{\zeta}=\zeta-\zeta^2+\varepsilon\zeta^3 g(\zeta,\varepsilon)$$

with a smooth function g. The corresponding first integral now reads

$$I_\varepsilon(\zeta,\dot\zeta)=\frac{1}{2}\dot\zeta^2-\frac{\zeta^2}{2}+\frac{\zeta^3}{3}+\varepsilon\zeta^4 G(\zeta,\varepsilon)$$

with a smooth function $G(\zeta,\dot\zeta)$. In order to prove the nonlinear orbital stability we apply again the moment of instability. We obtain

$$\begin{aligned} m(\varepsilon) &= \int_{-\infty}^{\infty} v'^2\,d\xi \\ &= \varepsilon^{5/2}\int_{-\infty}^{\infty} -\zeta\bigl(\zeta-\zeta^2+\varepsilon\zeta^3 g\bigl(\zeta(z),\varepsilon\bigr)\bigr)\,dz \\ &=: \varepsilon^{5/2}A(\varepsilon). \end{aligned}$$

We differentiate twice to get

$$m''(\varepsilon)=\frac{15}{4}\varepsilon^{1/2}A(\varepsilon)+5\varepsilon^{3/2}A'(\varepsilon)+\varepsilon^{5/2}A''(\varepsilon).$$

Since A is continuous on $(0,\varepsilon_1)$, it remains to show that $A'(\varepsilon)$ and $A''(\varepsilon)$ are bounded independently of ε. Smooth dependence of ε and exponential decay of $\zeta(z)$, and its derivatives w.r.t. ε follows from the ODE of the profile with ε as parameter; this implies that the leading term for $\varepsilon\to 0$ is $15/4\varepsilon^{1/2}A(\varepsilon)$. The limit $A(0)$ though is the moment of instability of a soliton of the KdV equation with $p(v)=v^2$ and we know [4] that

$$A(0)>0. \qquad \square$$

Theorem 3a.

(i) *The conclusion of Theorem* 3 *holds also if* $p''(v_*)=0$ *and either*

$$p'''(v_*)>0$$

or

$$p'''(v_*)=0 \quad \textit{and} \quad p''''(v_*)>0.$$

(ii) *If, however,*

$$k:=\min_{j\geqslant 2}\bigl\{p^{(j)}(v_*)\neq 0\bigr\}>5,$$

then there exists an $\varepsilon_1>0$ *such that for all* $\varepsilon\in(0,\varepsilon_1)$ *there exists a small amplitude solitary wave and this wave is not stable.*

Proof. This fact is related to the transition from stability to instability for power laws $p(v)=kv^q$ from $q\leqslant 4$ to $q>5$. For this dichotomy see [4]. (i) W.l.o.g. we assume again that $p(v)=v^q+O(v^{q+1})$ with $q\in\{3,4\}$. Adopting the notation from the above proof, the scaling

$$v(\xi):=\varepsilon^{1/(q-1)}\zeta\bigl(\varepsilon^{1/2}\xi\bigr)$$

transforms the corresponding profile equations into

$$\ddot{\zeta} = \zeta - \zeta^q + \varepsilon^{1/(q-1)}\zeta^{q+1} g(\zeta, \varepsilon).$$

Again, the $\varepsilon \to 0$ limit object is a stable KdV soliton which shows the stability for small ε.

(ii) The finite-size solitary wave corresponding to $\varepsilon = 0$ is an unstable KdV soliton. Its moment of instability is concave, and the linearization has an eigenvalue in the right half-plane; cf. [9]. This property is robust. □

4. (In-)Stability of small amplitude solitary waves in the Euler–Korteweg equation

The Euler–Korteweg equations describe the motion of a compressible, inviscid fluid with a variable internal capillarity. They have been studied intensively in the last years, particularly by Benzoni-Gavage and co-authors; see [2,1] and references therein. In one space dimension and a Lagrangian coordinate y, the one-dimensional isothermal model has the form

$$\begin{aligned} V_t - U_y &= 0, \\ U_t + p(V)_y &= -\left(\kappa(V)V_{yy} + \frac{1}{2}\big(\kappa(V)\big)_y V_y\right)_y \end{aligned} \tag{11}$$

where U, V, and $p(V)$ denote velocity, specific volume, and pressure of the fluid, and $\kappa(V) > 0$ its volume-dependent capillarity. The special case $\kappa \equiv 1$ is the generalized Boussinesq equation.

In [2] the authors were mainly interested in nonlinear waves in so-called van der Waals fluids, i.e. media exhibiting phase changes. We will here concentrate on the easier situation of a classical monotone, convex pressure function, i.e., we only assume that at a reference value v_* of the specific volume,

$$p'(v_*) < 0 \quad \text{and} \quad p''(v_*) > 0. \tag{12}$$

The equations governing the profile $(v, u)(\xi)$ connecting $(v_*, 0)$ read

$$\begin{aligned} -cv' &= u', \\ -cu' + p(v)' &= -\left(\kappa(v)v'' + \frac{1}{2}\big(\kappa(v)\big)' v'\right)'. \end{aligned} \tag{13}$$

This equation is equivalent to

$$\kappa(v)v'' + \frac{1}{2}\big(\kappa(v)\big)' v' = -p(v) + p(v_*) - c^2(v - v_*) \tag{14}$$

and possesses (cf. [1]) a first integral given by

$$I\big(v, v'\big) = \frac{1}{2}\kappa(v)v'^2 - f(v) + f(v_*) - p(v_*)(v - v_*) + \frac{1}{2}c^2(v - v_*)^2, \tag{15}$$

with f the classical part of the free energy,

$$-\frac{df(v)}{dv} = p(v).$$

We can adapt Definition 2 for the Euler–Korteweg equation; note, however, that it is natural to define stability independently of a requirement for temporally global existence; cf. [1]. Our main result is

Theorem 4. *Consider* (11) *with smooth functions* p *satisfying* (12)*. Then*:

(i) *There is a* $c_1 < c_* := \sqrt{-p'(v_*)}$ *such that for each* $c \in (c_1, c_*)$ *there exists a small amplitude solitary wave solution with profile* (v_c, u_c)*. The map* $c \mapsto (v_c, u_c)$ *is* $C^\infty((c_1, c_*))$.
(ii) *There is a* $c_2 \in (c_1, c_*)$ *such that for each* $c \in (c_2, c_*)$ *the solitary wave* (v_c, u_c) *is orbitally stable.*

Proof. Shifting and scaling v and p, we assume without loss of generality that

$$v_* = 0 \quad \text{and} \quad p(v) = -v + v^2 + O\left(v^3\right) \quad (\text{and thus } c_* = 1).$$

(i) Let $\varepsilon := 1 - c^2$ and $z := \varepsilon^{1/2}\xi$. The scaling

$$v(\xi) =: \varepsilon\zeta(z) \tag{16}$$

transforms (14) into

$$\kappa(\varepsilon\zeta)\ddot{\zeta} = -\frac{1}{2}\frac{d(\kappa(\varepsilon\zeta))}{dz}\dot{\zeta} + \zeta - \zeta^2 + \varepsilon\zeta^3 g(\zeta, \varepsilon), \tag{17}$$

with a smooth function g. The first integral (15) now reads

$$I_\varepsilon(\zeta, \dot{\zeta}) = \frac{1}{2}\kappa(\varepsilon\zeta)\dot{\zeta}^2 - \frac{1}{2}\zeta^2 + \frac{1}{3}\zeta^3 + \varepsilon\zeta^4 G(\zeta, \varepsilon)$$

with a smooth function G. A way to prove the existence of homoclinic orbits for small ε is to show that

$$F(\zeta, \varepsilon) := I_\varepsilon(\zeta, 0)$$

vanishes for $\zeta > 0$. As

$$F\left(\frac{3}{2}, 0\right) = 0 \quad \text{and} \quad \frac{\partial F}{\partial \zeta}\left(\frac{3}{2}, 0\right) \neq 0,$$

the implicit function theorem shows that there is a smooth curve $\zeta = \zeta_0(\varepsilon), 0 \leqslant \varepsilon < \varepsilon_1$ along which $F(\zeta_0(\varepsilon), \varepsilon) = 0$. Recalling that $\varepsilon = 1 - c^2$, we see that the first sentence of (i) is proved. As the homoclinic orbit connects a hyperbolic fixed point to itself, the solution (v, u) together with its derivatives tends exponentially fast to zero as $z \to \infty$. Smooth dependence of c follows by a standard ODE argument.

(ii) In order to show nonlinear orbital stability, we apply again the moment of instability that was first used for the Euler–Korteweg equations in [2]. The moment of instability is defined in the same way as for the generalized Boussinesq equation. The Hamiltonian though now reads

$$H(W) = \frac{1}{2}U^2 + f(V) + \frac{1}{2}\kappa(V)V_y^2,$$

and the momentum simplifies to

$$I(W) = UV.$$

Theorem (∗) for the Euler–Korteweg equation is due to [2]. As assumptions (A1) and (A3) are fulfilled for the Euler–Korteweg equation [1,2] and as statement (i) of our theorem just means (A2) with $(c_-, c_+) = (c_1, 1)$, statement (ii) will be proved once we have verified the convexity of $m(c)$.

Using the identities

$$0 = I\big(v, v'\big) - I(0,0) = \frac{1}{2}\kappa(v)\big(v'\big)^2 - f(v) + \frac{1}{2}c^2 v^2$$

and $u = -cv$ we obtain

$$\begin{aligned} m(c) &= \int_{-\infty}^{\infty} (\mathcal{H} + c\mathcal{I})(v,u)\,dy \\ &= \int_{-\infty}^{\infty} \frac{1}{2}u^2 + f(v) + \frac{1}{2}\kappa(v)(v_y)^2 + cvu\,dy \\ &= \int_{-\infty}^{\infty} \kappa(v)(v_y)^2\,dy. \end{aligned}$$

Integration by parts, the scaling (16), and Eq. (17) yield

$$\begin{aligned} m(c) &= \varepsilon^3 \int_{-\infty}^{\infty} \kappa\big(\varepsilon\zeta\big(\varepsilon^{1/2}\xi\big)\big)\big(\dot\zeta\big(\varepsilon^{1/2}\xi\big)\big)^2\,d\xi \\ &= \varepsilon^{\frac{5}{2}} \int_{-\infty}^{\infty} -\zeta(z)\bigg(\frac{d\kappa(\varepsilon\zeta(z))}{dz}\dot\zeta(z) + \kappa\big(\varepsilon\zeta(z)\big)\ddot\zeta(z)\bigg)\,dz \\ &= \varepsilon^{\frac{5}{2}} \int_{-\infty}^{\infty} -\bigg(\frac{1}{2}\frac{d\kappa(\varepsilon\zeta(z))}{dz}\dot\zeta(z)\zeta(z) + \zeta^2(z) - \zeta^3(z) + \varepsilon\zeta^4(z)g\big(\zeta(z),\varepsilon\big)\bigg)\,dz. \end{aligned}$$

Recall that $\varepsilon(c) = 1 - c^2$. Let $A(c)$ denote the integral term in the last line, i.e. $m(c) = -\varepsilon^{5/2}A(c)$. To see that $m(c)$ is convex for $c \to 1$, we differentiate twice to get

$$m''(c) = \big(+15\big(1-c^2\big)^{1/2}c^2 - 5\big(1-c^2\big)^{3/2}\big)A(c) - 10\big(1-c^2\big)^{3/2}A'(c) + \big(1-c^2\big)^{5/2}A''(c)$$

whose leading term for $c \to 1$ is $15(1-c^2)^{1/2}c^2 A(c)$. As $A(1)$ is the moment of instability of a finite-amplitude soliton for the Boussinesq equation, we know [3] that

$$A(1) > 0. \tag{18}$$

Since A is continuous on $(c_1, 1]$, it remains to show that $A'(c)$ and $A''(c)$ are bounded independently of c. This follows however directly from the following observation.

Lemma 8. *The scaled profile $\zeta(z)$ and its derivatives $\partial_c\zeta(z)$, $\partial_c^2\zeta(z)$ with respect to c decay to zero exponentially for $z \to \infty$.*

Proof. Let $\eta := \frac{d\zeta}{d\varepsilon}$ and consider

$$\begin{aligned}
\kappa(\varepsilon\zeta)\ddot{\zeta} &= -\frac{1}{2}\kappa'(\varepsilon\zeta)(\dot{\zeta})^2 + \zeta + \zeta^2 + \varepsilon\zeta^3 g(\zeta,\varepsilon),\\
\kappa(\varepsilon\zeta)\ddot{\eta} &= -\frac{\kappa'(\varepsilon\zeta)(\zeta+\varepsilon\eta)}{\kappa(\varepsilon\zeta)}\left(-\frac{1}{2}\kappa'(\varepsilon\zeta)(\dot{\zeta})^2 + \zeta + \zeta^2 + \varepsilon\zeta^3 g(\zeta,\varepsilon)\right)\\
&\quad -\frac{1}{2}\kappa''(\varepsilon\zeta)(\zeta+\varepsilon\eta)(\dot{\zeta})^2 - \kappa'(\varepsilon\zeta)\dot{\zeta}\dot{\eta} + \eta - 2\zeta\eta + O(\zeta^2),\\
\dot{\varepsilon} &= 0.
\end{aligned}$$

Its linearization at the fixed point $(\zeta, \dot{\zeta}, \eta, \dot{\eta}) = (0,0,0,0)$ has eigenvalues $\lambda = \pm\kappa(0)^{-1/2}$ and $\lambda = 0$ which shows exponential decay of ζ and η for $z \to \infty$ as well as smooth dependence of ε, and hence c. The same argument applies to $\partial_\varepsilon^2\zeta$. $\square$

Theorem 4a.

(i) *The conclusion of Theorem* 4 *holds also if $p'(v_*) < 0$ and $p''(v_*) = 0$ and either*

$$p'''(v_*) > 0$$

or

$$p'''(v_*) = 0 \quad \textit{and} \quad p''''(v_*) > 0.$$

(ii) *If, however, $p'(v_*) < 0$ and*

$$k := \min_{j\geqslant 2}\bigl\{p^{(j)}(v_*) \neq 0\bigr\} > 5,$$

then assertion (i) *of Theorem* 4 *again holds, but there now is a $c_2 \in (c_1, c_*)$ such that for each $c \in (c_2, c_*)$ the solitary waves (ϕ_c, ψ_c) are not stable.*

Proof. (i) W.l.o.g. we assume again that $p(v) = -v + v^q + O(v^{q+1})$ with $q \in \{3,4\}$. Adopting the notation from the above proof, the scaling

$$\phi(\xi) := \varepsilon^{1/(q-1)}\zeta\bigl(\varepsilon^{1/2}z\bigr)$$

transforms the corresponding profile equations into

$$\kappa\bigl(\varepsilon^{1/(q-1)}\zeta\bigr)\ddot{\zeta} = -\frac{1}{2}\frac{d(\kappa(\varepsilon^{1/(q-1)}\zeta))}{dz}\dot{\zeta} + \zeta - \zeta^q + \varepsilon^{1/(q-1)}\zeta^{q+1}g(\zeta,\varepsilon).$$

Again, the $\varepsilon \to 0$ limit object is a stable Boussinesq soliton which shows the stability for small ε.

(ii) The finite-size solitary wave corresponding to $c = 1$ is an unstable Boussinesq soliton. Its moment of instability is partly concave, and the linearization has an eigenvalue in the right half-plane; cf. [10] (see also [8]). This property is robust. $\square$

5. Remarks

1. As an obvious consequence of our small-amplitude results, we note that no constant state solution

$$\big(v(y,t), u(y,t)\big) = (v_*, u_*) \quad \text{with (12)}$$

can be time-*asymptotically* stable towards perturbations which have the form of a small-amplitude soliton.

2. Obviously, there are well-known examples for flux functions $p(v)$ for the gKdV equation as well as for the generalized Boussinesq equation which are convex but whose first derivative is not concave and whose solitary waves are not stable, notably

$$p(v) = v^q \quad \text{with } q > 5 \text{ in the gKdV-case (cf. [4,9])}$$

and

$$p(v) = -v + v^q \quad \text{with } q > 5 \text{ for the generalized Boussinesq equation (cf. [10]).}$$

3. Theorem 3 follows actually from Theorem 4. Let us briefly sketch this idea: We connect the stability of waves in the gKdV equation with that of waves in the generalized Boussinesq equation. To make this nice little connection, we start by observing that the moment of instability only depends on the profile equation. Let w.l.o.g. $p(v) = -v + v^q + O(v^{q+1})$ and $v_* = 0$. Denote by $m^B(c)$ resp. $m^K(\varepsilon)$ the moment of instability for solitary waves in the generalized Boussinesq resp. gKdV equations. They satisfy

$$m^B(c) = \int_{-\infty}^{\infty} \big((\phi_c^B)'\big)^2 \, d\xi$$

and

$$m^K(\varepsilon) = \int_{-\infty}^{\infty} \big((\phi_\varepsilon^K)'\big)^2 \, d\xi$$

where ϕ_c^B resp. ϕ_ε^K are respective profiles of the generalized Boussinesq and gKdV equations, solving

$$\big(\phi_c^B\big)'' = \big(1 - c^2\big)\phi_c^B - \big(\phi_c^B\big)^q - h\big(\phi_c^B\big)$$

resp.

$$\big(\phi_\varepsilon^K\big)'' = (1 + \varepsilon)\phi_\varepsilon^K - \big(\phi_\varepsilon^K\big)^q - h\big(\phi_\varepsilon^K\big)$$

with $h(v) = p(v) + v - v^q$. The profiles differ only in their dependence on the speed; more precisely, we have

$$m^B(c) = m^K\big(g(c)\big) \quad \text{with } g(c) = -c^2$$

and thus

$$\left(m^K\right)''\left(g(c)\right) = \frac{(m^B)''(c)}{4c^2} - \frac{(m^B)'(c)}{4c^3}.$$

We investigate the behavior of $(m^K)''(g(c))$ for $c \to p'(v_*) = 1$. Using the notation from the proof of Theorem 4, with $\kappa \equiv 1$, we find

$$\left(m^K\right)''\left(-c^2\right) = \frac{15\sqrt{1-c^2}}{4} A(c) + O\left(\left(1-c^2\right)^{3/2}\right).$$

Due to (18) and Theorem 4, this implies in the case $q \leqslant 4$ that $m(\varepsilon)$ is convex for sufficiently small ε. In the case $q > 5$ $A(1)$ is negative and we can conclude instability.

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